Axiom V_3. *Any point and direction issuing from it may be moved into any desired point and any desired direction issuing from that point—and the correspondence of these elements completely and uniquely determines the rigid motion.**

Definition. *Geometrical figures which may be carried over into one another by rigid motions are said to be congruent. The sign for congruence consists of three horizontal parallel strokes*, as $\equiv$.

$$A \text{ and } a \equiv A' \text{ and } a'.$$

POLAR COORDINATE PROOFS OF TRIGONOMETRIC FORMULAS.

By OSWALD VEBLEN, The University of Chicago.

1. Graphical, that is to say analytic geometrical methods, seem at present to be on the gain in the teaching of Trigonometry. Particularly true is this in courses conducted by the "LaboratoryMethod." This fall, I have obtained rather pleasing results by adopting a suggestion of Professor Moore to use polar coördinates. The geometric simplicity of these graphs, the sine and cosine being represented by circles and the secant and cosecant by straight lines (see Fig. 1), not only makes them attractive to the student but, unlike theCartesian graphs, makes them useful in proving theorems.

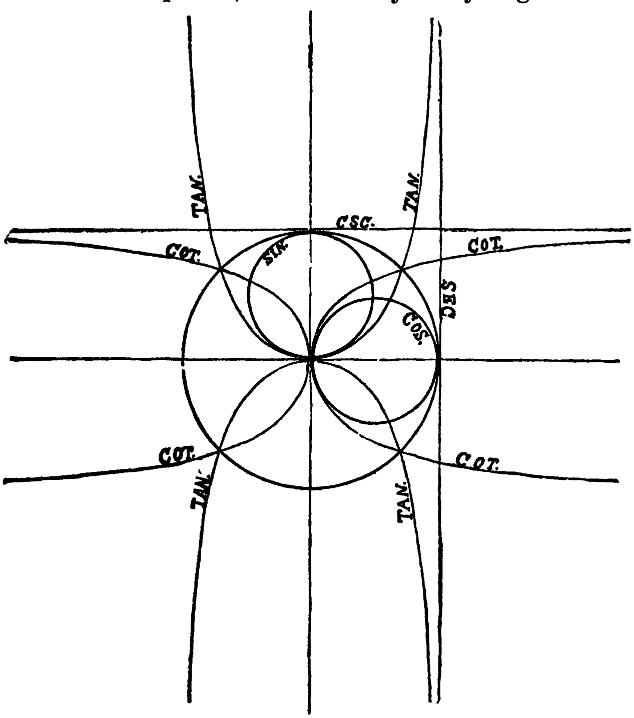


Fig. 1.

The proofs† given below, it is hoped, will demonstrate this latter point.

* The reader should convince himself that in case change of size or shape is allowed the correspondence of A and a to A' and a' will not be sufficient completely to determine the motion. The amount of distortion must also be somehow specified.

† While these proofs are probably to be found somewhere in the literature, I have not been able to find them.

That they contain some elements of simplicity I am convinced by the fact that several of my students worked out the proof of the formula for $\sin(x+y)$ with no other help than the mere suggestion to use polar coördinates. §§6, 7, 8 are based on memoranda given me by Professor Moore of work intended for his elementary calculus course. The proofs are made only for positive angles less than $\frac{1}{2}\pi$.

2. *If an angle is inscribed in a circle of unit diameter its sine is the chord of the arc subtended.*

If one side OB of the angle AOB is a diameter of the circle (see Fig. 2), then since OBA is a right angle, $\dfrac{AB}{OA}=\dfrac{AB}{1}$ is the sine of AOX. If the angle is inscribed in any other way, by a familiar theorem, it subtends the same chord as AOB.

The theorem is also true in the limiting case where one side of the angle

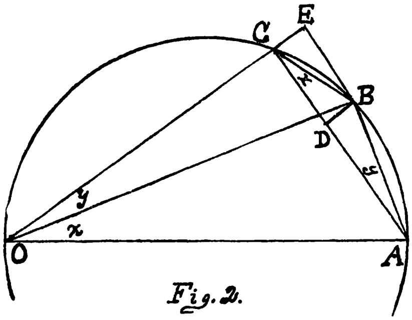


Fig. 2.

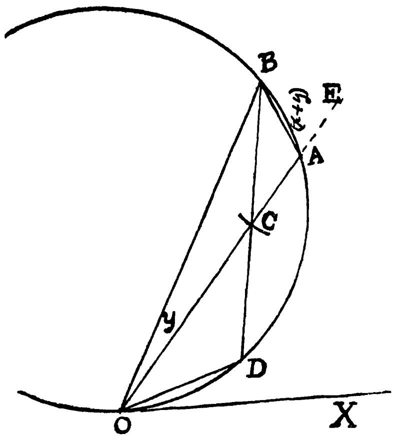


Fig. 3.

is tangent to the circle. This is the polar coördinate case and thus, in Fig. 3, $OA=\sin AOX$. We may note also that in a circle of unit diameter the length of the arc subtended by an inscribed angle is the measure of that angle in radians.

§3. Proof of the formula $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

In a circle ABO (Fig. 2)* of unit diameter, let

$$\begin{array}{ll} \angle AOB=x, & \therefore AB=\sin x. \\ \angle BOC=y, & \therefore BC=\sin y. \\ \therefore \angle AOC=x+y, & \therefore AC=\sin(x+y). \end{array}$$

Let BD be perpendicular to AC. Then

$$\angle BAC=y \text{ (subtending same arc as } \angle BOC)$$
$$\angle BCA=x \text{ (subtending same arc as } \angle AOB).$$

*Of course, in this proof OA need not be a diameter.

$\therefore AD = AB\cos y = \sin x\cos y,$
$DC = BC\cos x = \sin y\cos x.$
Since $AC = AD + DC$, $\sin(x+y) = \sin x\cos y + \cos y\sin x$.

This proof applies directly to the polar graph (see Fig. 4) if OA is taken tangent to the circle (so that $A = O$). The present form is intended to suggest its use to those who do not care to introduce polar coördinates in a beginning course.

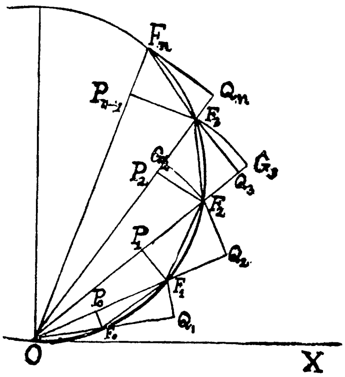


Fig. 4.

4. Proof that $\cos(x+y) = \cos x\cos y - \sin x\sin y$.

Let OAB (Fig. 2) be the polar cosine curve, *i. e.*, OA is the diameter $(=1)$ of the circle OAB.

$$OB = \cos x, \quad OC = \cos(x+y), \quad BC = \sin y.$$

Let BE be perpendicular to OC. $\therefore OE/OB = \cos y$. $\therefore OE = \cos x\cos y$.

EB and CA are parallel, both being perpendicular to OC. $\therefore \angle CBE = \angle BCA = x$. $\therefore CE/CB = \sin x$. $\therefore CE = \sin x\sin y$.

Since $OC = OE - CE$, $\cos(x+y) = \cos x\cos y - \sin x\sin y$.

5. Proof that $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$.

In the polar sine circle, Fig. 3, $OA = \sin x$, $AB = \sin y$. On OB, lay off $AC = AB$ and let D be the point in which BC meets the circle.

$\therefore OC = \sin x - \sin y$.

By elementary geometry, $\angle BAE = \angle AOB + \angle ABO = x + y$.

$\therefore \angle CBA = \angle BCA = \frac{x+y}{2}$. $\quad \therefore \angle OCD = \frac{x+y}{2}$.

Now $\angle COD$ is measured by the arc AD and hence $\angle COD = \angle ABD = \frac{x+y}{2}$

Hence OCD is an isosceles triangle and $OC = 2.OD\cos\frac{x+y}{2}$.

But $OD = \sin(\angle OBD) = \sin\left(x - \frac{x+y}{2}\right) = \sin\frac{x-y}{2}$.

$\therefore \sin x - \sin y = OC = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$.

§6. First proof of the formulas

$$D_x\sin x = \cos x \quad\ldots\ldots\ldots\ldots (1).$$

$$\int_{x_0}^{X}\cos x\,dx = \sin X - \sin x_0 \quad\ldots\ldots\ldots\ldots (2).$$

The proofs in this section make use of Fig. 4, in which $\angle XOF_0 = x_0$; $\angle F_0OF_1 = \triangle x = \angle F_{i-1}OF_i$ $(1 = 1 \ldots n)$; or, if one prefers to speak of the arcs,

$$\triangle x = \text{arc}F_0F_1 = \text{arc}F_1F_2 = \ldots = \text{arc}F_{n-1}F_n.$$

$F_{i-1}P_{i-1}$ is perpendicular to OF_i and F_iQ_i is perpendicular to OF_{i-1}. To prove (1), we make use of only one of the $\triangle x$ portions of the figure, for example, the third. OG_2 is taken equal to OF_2 and $OG_3=OF_3$. Then by elementary geometry

$$P_2F_3 > G_2F_3 = G_3F_2 > F_2Q_3 \quad\ldots\ldots\ldots\ldots(3).$$

But if we call $\angle XOF_2=x$, $\angle F_2F_3O=x$ and $\angle F_3F_2Q_3=x+\triangle x$,

$$\begin{aligned}
F_2G_3 &= OF_3-OF_2=\sin(x+\triangle x)-\sin x,\\
P_2F_3 &= F_2F_3\cos x=\sin\triangle x\ \cos x,\\
F_2Q_3 &= F_2F_3\cos(x+\triangle x)=\sin\triangle x\ \cos(x+\triangle x).
\end{aligned}$$

Hence (3) says that

$$\sin\triangle x\ \cos x > \sin(x+\triangle x)-\sin x > \sin\triangle x\ \cos(x+\triangle x)\quad\ldots\ldots\ldots(4).$$

$$\therefore\ \frac{\sin\triangle x}{\triangle x}\cos x > \frac{\sin(x+\triangle x)-\sin x}{\triangle x} > \frac{\sin\triangle x}{\triangle x}\cos(x+\triangle x).$$

Since $\underset{h=0}{L}\dfrac{\sin h}{h}=1$ and $\cos x$ is continuous, both extremes of this double inequality approach $\cos x$ as $\triangle x$ approaches zero.

Therefore the middle term approaches $\cos x$ and we have

$$D_x\sin x=\underset{\triangle x=0}{L}\frac{\sin(x+\triangle x)-\sin x}{\triangle x}=\cos x.$$

In this, according to our figure, $\triangle x$ was always positive. But if XOF_3 had been taken as x the same figure with similar reasoning would prove (4) for that case also.

Of course the theorem that for continuous functions, integration is the inverse of differentiation shows that (2) is a corollary of (1). But for some purposes of instruction it is worth while to compute (2) directly from the definition of an integral as the limit of a sum. Assuming the existence of a definite integral for $\cos x$ we have

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=1}^{n}\cos(x_0+k\triangle x)\triangle x\quad\ldots\ldots\ldots(5).$$

where $\triangle x=(X-x_0)/n$, and also*

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=0}^{n-1}\cos(x_0+k\triangle x)\triangle x\quad\ldots\ldots\ldots(6).$$

*In the familiar Cartesian figure, (5) corresponds to the inner set of rectangles and (6) to the outer.

Since $\underset{\triangle x=0}{L} \dfrac{\sin\triangle x}{\triangle x}=1$, (5) and (6) can be replaced by (7) and (8):

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=1}^{n}\cos(x_0+k\triangle x)\sin\triangle x=\underset{n=\infty}{L}S_n\text{.....}(7),$$

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\sum_{k=0}^{n-1}\cos(x_0+k\triangle x)\sin\triangle x=\underset{n=\infty}{L}S'_n\text{.....}(8),$$

and (2) will be proved if we show that $S_n<\sin X-\sin x<S'_n$.

From the quadrilateral $F_0P_0F_1Q_1$, we obtain, as we obtained (4),

$$\sin\triangle x\cos x_0>\sin(x_0+\triangle x)-\sin x_0>\sin\triangle x\cos(x_0+\triangle x).$$

From the second quadrilateral $F_1P_1F_2Q_2$ we similarly get

$$\sin\triangle x\cos(x_0+\triangle x)>\sin(x_0+2\triangle x)-\sin(x_0+\triangle x)>\sin\triangle x\cos(x_0+2\triangle x),$$

and so on. From the last quadrilateral we obtain, calling $X<XOF_n=x_0+n\triangle x$,

$$\sin\triangle x\cos[x_0+(n-1)\triangle x]>\sin X-\sin[x_0+(n-1)\triangle x]>\sin\triangle x\cos(x_0+n\triangle x).$$

Adding together these inequalities we see that the sum of the first terms is S'_n, and of the last terms is S_n. In the middle terms everything else cancels, leaving only $\sin X-\sin x_0$. So we have as we desired

$$S'_n>\sin X-\sin x_0>S_n.$$

This result can be seen still more directly by noting that

$$S'_n=P_0F_1+P_1F_2+\text{.....}+P_{n-1}F_n,$$
$$S_n=F_0Q_1+F_1Q_2+\text{.....}+F_{n-1}Q_n,$$
$$OF_0=\sin x_0,\quad OF_n=\sin X.$$

If the rays centering at O be imagined to fold together like a fan from OF_0 to OF_n it is evident that S_n is less and S'_n greater than OF_n-OF_0.

§7. Second Proof of (1) and (2).

In some quarters there is a tendency to reverse the old order and present the integral calculus before the differential. The definitions of the two operations of differentiation and integration are certainly independent of each other; and whatever order may be preferred for pedagogical reasons, it is not amiss to see that in either case precisely similar methods can be used in deriving the formulas for the usual functions. That such is the case depends depends on the following theorem.*

*The proof of the first part of this theorem is made possible by the fact that any monotonic function is integrable,—a monotonic function being such that if $a<b$ either always $f(a)>$ or $=f(b)$ or always $f(a)<$ or $=f(b)$. Just as we did for the special case of $\sin x$ in the last section we can let $b-a=\triangle x$ and add up n inequalities like (9) and thus have $S_n>F(x)-Fx>S'n$. To prove the second part divide by $b-a$ and pass to the limit as b approaches a.

If on an interval x_0, ..., X, two functions $f(x)$ and $F(x)$ have the property that for every two values of x, a and b $(x_0 \leqq a < b \leqq X)$,

$$f(a)(b-a) > F(b)-F(a) > f(b)(b-a) \quad \text{..........................} (9)$$

then, first, $\int_{x_0}^{X} f(x)dx = F(X) - F(x_0)$; *second, if $f(x)$ is continuous,* $D_xF(x) = f(x)$.

In view of this theorem, to prove (1) and (2) we need only to prove the inequality

$$\cos x_0.(x-x_0) > \sin x - \sin x_0 > \cos x(x-x_0), \quad 0 \leqq x_0 < x \leqq \pi/2 \text{.....} (10).$$

To this end we make use of the inner part of Fig. 5 in which

$$\angle XOS_0 = x_0, \quad \angle XOS = x.$$

S_0U is perpendicular to OS, S_0V to OS_0, and SO'' and S_0O'' are tangents to the sine curve OS_0B. About O'' a circle is described with radius $O''S = O''S_0$ and meeting $O''S$ in W. Since

$$\angle S_0VS = \tfrac{1}{2}\pi + (x-x_0)$$
$$= \pi - [\tfrac{1}{2}\pi - (x-x_0)],$$

and $\angle S_2O''S = \pi - 2(x-x_0)$

$$= 2[\tfrac{1}{2}\pi - (x-x_0)],$$

the circle about O'' must pass through V. Hence $SVW = \tfrac{1}{2}\pi$. From these considerations it follows that

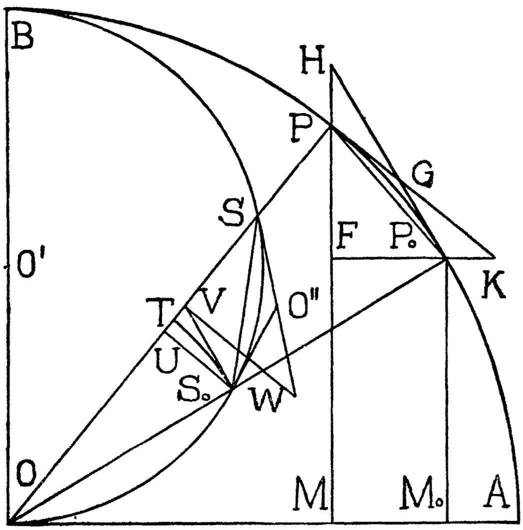


Fig. 5.

$$\sin x - \sin x_0 = TS > VS = \cos VSW.WS$$
$$= \cos x.(S_0O'' + O''S) > \cos x.\mathrm{arc}S_0S = \cos x.(x-x_0),$$

$$\text{and } TS < US = \cos USS_0.S_0S < \cos x_0.\mathrm{arc}S_0S < \cos x_0(x-x_0),$$

which proves (10).

§8, Second Proof of (1) and (2).

Without going into details I will add the outline of a second proof of Professor Moore's for the inequality (10) and hence for (1) and (2). This, unlike the others, is not a polar coördinate proof, but uses the unit circle. In the outer part of Fig. 5,

$$\sin x - \sin x_0 = FP = \cos FPK.PK > \cos x.(PG + GP_0) > \cos x.\mathrm{arc}\ P_0P = \cos x.(x-x_0),$$

$$FP=\cos FPP_0.P_0P=\cos\frac{x+x_0}{2}.P_0P<\cos x_0P_0P<\cos x_0(x-x_0).$$

$\therefore \cos x.(x-x_0)<\sin x-\sin x_0<\cos x_0(x-x_0)$, which is (10).
The outer part of Fig. 5 can also be used to prove that

$$\sin x-\sin x_0=2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}.$$

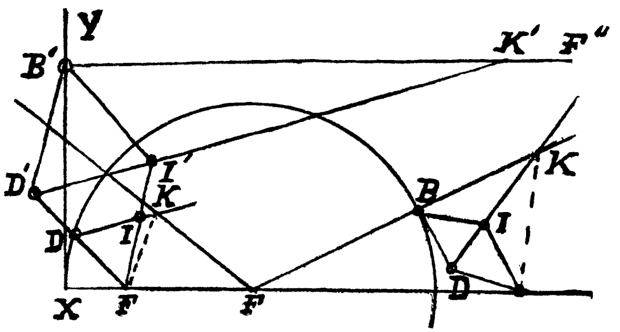
A LINKAGE FOR DESCRIBING THE CONIC SECTIONS BY CONTINUOUS MOTION.

By JOHN JAMES QUINN, Ph. B., Head of the Department of Mathematics and Manual Training, Warren High School, Warren, Pa.

The linkage is a material embodiment of the facts and conditions setforth in the following

THEOREM: *If one vertex of a movably pivoted rhombus is constrained to move in the circumference of a directing circle, while the opposite vertex is fixed in the diameter (or diameter produced), the locus of the intersection of the diagonal (produced) through the other two vertices with the radius of the directing circle is a conic.*

Let $BIDF$ be the rhombus with the vertex B moving in the circumference of the directing circle whose center is F'; F the opposite vertex fixed within (without) the diameter; and DI the diagonal produced to intersect the radius (produced) in the point K. Draw KF, and FK'. Then in the figure to the left,



$$FK+KF'=BK+KF'=BF'=\text{constant}.\quad \therefore \text{The locus of } K \text{ is an ellipse.}$$

In the figure to the right,

$$F'K-FK=F'K-BK=BF'=\text{constant}.\quad \therefore \text{The locus of } K \text{ is an hyperbola.}$$

Now suppose that the radius of the directing circle becomes infinite. Then the circumference becomes the line XY, perpendicular to the diameter (the directrix); BF becomes the line $B'F''$, parallel to the diameter, and the diagonal $D'I'$ intersects it in K'.